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Exactly solvable models of interacting spin-*s* particles in one dimension

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Abstract
We consider the eigenspectrum solution of a many-body problem of interacting spin-s particles that can be solvable within the generalized Bethe ansatz method. We assume that the interactions are encoded in terms of an arbitrary $U(1)$ invariant factorizable S-matrix. The exact solution of the spin part is based on a unified formulation of the quantum inverse scattering method for an arbitrary $(2s+1)$-dimensional monodromy matrix. The respective eigenstates are shown to be given in terms of $2s$ creation fields by a general new recurrence relation. This allows us to derive the spectrum and the respective Bethe ansatz equations.

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The existence of exactly solved models has been playing a fundamental role in our understanding of interacting one-dimensional many-particle systems in distinct areas of theoretical physics [1–4]. An important class of models is N-particle systems whose state vector has the structure of a generalized Bethe ansatz wavefunction [5]. In these cases, the state of a given jth particle carries besides momenta $k(\mu_j)$ and energy $\epsilon(\mu_j)$, parameterized by some rapidity μ_j, an extra quantum number $a_j = 1, \dots, n_s$ which represents the many possible n_s species of particles. The Bethe ansatz solution asserts that the projection of the state vector of the system in the region the particle coordinates are ordered as $0 \leqslant x_{Q_1} < x_{Q_2} < \cdots < x_{Q_N} \leqslant L$ has the form [2, 4, 5],

$$\Psi_{a_1 \dots a_N}(x_1, \dots, x_N) = \sum_P A_{a_1,\dots,a_N}(P|Q) \exp\left[\mathrm{i} \sum_{j=1}^{N} k(\mu_{P_j}) x_{Q_j} \right], \tag{1}$$

where $P \equiv \{P_1, \dots, P_N\}$ and $Q \equiv \{Q_1, \dots, Q_N\}$ denote permutations of the numbers $1, \dots, N$ that index the particles. The sum runs over all $N!$ permutations P of $\{1, \dots, N\}$ and the complex coefficients $A_{a_1,\dots,a_N}(P|Q)$ account for the wavefunction internal degrees of freedom.

The wavefunctions of the various regions, according to the possible ordering of the particles on the ring of size L, are expected to be connected to one another only through a sequence of two-body exchange processes. For instance, two distinct regions $(P|Q)$ and $(\bar{P}|\bar{Q})$ differing by the permutation of neighbouring ith and jth particles will be related by

$$A_{a_1,\ldots,a_i,a_j,\ldots,a_N}(\bar{P}|\bar{Q}) = S(\mu_i,\mu_j)_{a_i,a_j}^{b_i,b_j} A_{a_1,\ldots,b_i,b_j,\ldots,a_N}(P|Q), \tag{2}$$

where the elements $S(\mu_i,\mu_j)_{a_i,a_j}^{b_i,b_j}$ are supposed to encode the interactions between the particles. They are formally viewed as the amplitudes of the S-matrix of a factorizable scattering [6],

$$\hat{S}_{12}(\mu_1,\mu_2) = \sum_{a_1,a_2,b_1,b_2}^{n_s} S(\mu_1,\mu_2)_{b_1,b_2}^{a_1,a_2} e_{b_1,a_1} \otimes e_{b_2,a_2} \tag{3}$$

where $e_{a,b}$ denotes $n_s \times n_s$ Weyl matrices.

This type of factorizable theory has been first emerged in the study of the eigenspectrum problem associated with particles interacting through delta-function potential [5, 7]. It was then discovered that in order to assure exact solubility the scattering amplitudes needed to satisfy a necessary condition called Yang–Baxter equation [1, 5],

$$S(\mu_1,\mu_2)_{a_1,a_2}^{\gamma_1,\gamma_2} S(\mu_1,\mu_3)_{\gamma_1,a_3}^{b_1,\gamma_3} S(\mu_2,\mu_3)_{\gamma_2,\gamma_3}^{b_2,b_3} = S(\mu_2,\mu_3)_{a_2,a_3}^{\gamma_2,\gamma_3} S(\mu_1,\mu_3)_{a_1,\gamma_3}^{\gamma_1,b_3} S(\mu_1,\mu_2)_{\gamma_1,\gamma_2}^{b_1,b_2}, \tag{4}$$

where sum of repeated indices is assumed. Here we will also request that solutions of (4) are almost unitary, namely

$$S(\mu_1,\mu_2)_{a,b}^{c,d} S(\mu_2,\mu_1)_{d,c}^{\alpha,\gamma} = \rho(\mu_1,\mu_2)\delta_{a,\gamma}\delta_{b,\alpha}, \tag{5}$$

for some arbitrary function $\rho(\mu_1,\mu_2)$.

This framework has also been argued by Sutherland [8] to be of utility even when the wavefunction (1), (2) does not hold everywhere, as it did for the delta-function potential. The idea is that for some interactions with finite range R_c there exist regions $|x_i - x_j| \gg R_c$ where the particles behave as free ones and the off-mass-shell effects are expected to be neglected. The underlying scattering theory will then provide the conditions to match the wavefunction in adjacent free regions.

In this paper we assume the existence of such models that can either be solved exactly or asymptotically by the general Bethe wavefunction (1), (2). In any of the situations, one still has to find the quantization rule for the one-particle momenta $k(\mu_j)$, providing the means to study the physical properties of the system. In order to accomplish that one has to diagonalize the transfer matrix operator of an inhomogeneous vertex model of statistical mechanics whose Boltzmann weights are the elements of the non-diagonal factorized S-matrix (3). In what follows we will consider such a remaining problem in the general situation in which the scattering amplitudes depend on both values of two independent μ_1 and μ_2 rapidities. More specifically, when periodic boundary conditions are imposed on the wavefunction (1), (2), the one-particle momenta $k(\mu_j)$ are required to satisfy the following eigenvalue equation,

$$\mathrm{e}^{\mathrm{i}k(\mu_j)L} = \frac{\Lambda(\lambda=\mu_j,\overrightarrow{\mu})}{[\rho(\mu_j,\mu_j)]^{1/2}}, \qquad j=1,\ldots,N, \tag{6}$$

where $\overrightarrow{\mu}$ denotes the set $\{\mu_1,\mu_2,\ldots,\mu_N\}$ of rapidities. The energy E of the system is given by adding the one-particle energy expressions, namely

$$E = \sum_{j=1}^{N} \epsilon(\mu_j). \tag{7}$$

The function $\Lambda(\lambda,\overrightarrow{\mu})$ represents the eigenvalues of an auxiliary $[n_s]^N \times [n_s]^N$ operator $T(\lambda,\overrightarrow{\mu})$ usually called transfer matrix. This auxiliary eigenvalue problem can be defined by,

$$T(\lambda,\overrightarrow{\mu})|\psi\rangle = \mathrm{Tr}_{\mathcal{A}}[\mathcal{T}_{\mathcal{A}}(\lambda,\overrightarrow{\mu})]|\psi\rangle = \Lambda(\lambda,\overrightarrow{\mu})|\psi\rangle, \tag{8}$$

where the trace is taken over an auxiliary n_s-dimensional space $\mathcal{A} \equiv C^{n_s}$. Furthermore, the monodromy operator $\mathcal{T}_{\mathcal{A}}(\lambda, \vec{\mu})$ is related to the S-matrix elements by the following ordered product [9, 10],

$$\mathcal{T}_{\mathcal{A}}(\lambda, \vec{\mu}) = \hat{S}_{\mathcal{A}N}(\lambda, \mu_N)\hat{S}_{\mathcal{A}N-1}(\lambda, \mu_{N-1}) \ldots \hat{S}_{\mathcal{A}1}(\lambda, \mu_1). \tag{9}$$

In order to make further progress it becomes crucial the exact solution of the eigenvalue problem (8), (9) for arbitrary n_s and two-body scattering amplitudes. This is indeed a tantalizing open problem, especially when a particular form of $S(\mu_1, \mu_2)_{a,b}^{c,d}$ is not specified. Indeed, most of the results concentrate on specific S-matrices such as those related to the six-vertex model [11], to its higher spin descendents [12] and those based on higher rank Lie algebras [13–17]. In the latter case, some of the findings [16, 17] are still in the form of conjectures for the transfer matrix eigenvalues and also the number n_s actually encodes a variety of distinct conserved quantum numbers such as spin, colour, flavour, etc. Consequently, the number of null scattering coefficients grows rapidly with n_s due to the many possible underlying $U(1)$ symmetries.

In this work, however, we shall establish the essential tools to solve the eigenvalue problem (8), (9) when only a unique $U(1)$ symmetry is present for arbitrary n_s. This is the minimal continuous invariance one could request and our results will be valid for arbitrary factorizable S-matrices satisfying such symmetry condition and the unitarity relation (5). More precisely, we are considering integrable models whose S-matrices fulfil the property,

$$\left[\hat{S}_{12}(\mu_1, \mu_2), S_1^z + S_2^z\right] = 0, \tag{10}$$

where S_j^z is the azimuthal component of spin-s operator associated with the jth particle such that $s = (n_s - 1)/2$. Note that relation (10) means $S(\lambda, \mu)_{b_1,b_2}^{a_1,a_2} = 0$ unless the ice rule $a_1 + a_2 = b_1 + b_2$ is satisfied which leads us to a total number of $n_s\left(2n_s^2 + 1\right)/3$ non-null amplitudes.

The direct connection between the number of species and the spin of the particles makes these integrable models physically relevant. We shall tackle this problem by means of the quantum inverse scattering method [9, 10]. We remark that, in general, there is no known recipe to perform this task; therefore our results can be considered as new developments in this framework.

The most important quantity in this method is the monodromy matrix elements on the auxiliary space which here will be shortly denoted by $\mathcal{T}(\lambda)_{p,q}$, $p, q = 1, \ldots, n_s$. These are operators on the quantum space $\prod_{j=1}^{N} \otimes C_j^{n_s}$ such that the diagonal entries define the transfer matrix eigenvalue problem (8) while the off-diagonal ones will play the role of creation and annihilation fields. The commutation relations between such matrix elements are given with the help of the corresponding Yang–Baxter algebra,

$$S(\lambda, \mu)_{a,b}^{\alpha,\gamma} \mathcal{T}_{\alpha,p}(\lambda)\mathcal{T}_{\gamma,q}(\mu) = \mathcal{T}_{b,c}(\mu)\mathcal{T}_{a,d}(\lambda)S(\lambda, \mu)_{d,c}^{p,q}. \tag{11}$$

In order that the model be soluble by means of an algebraic Bethe ansatz, it is fundamental the existence a vacuum state $|0\rangle$ such that the monodromy operator (9) acts on it as a triangular matrix on the auxiliary space for arbitrary λ. Thanks to the underlying $U(1)$ symmetry, it is possible to build up this state by the tensor product of local vectors,

$$|0\rangle = \prod_{j=1}^{N} \otimes |s\rangle_j, \qquad S_j^z|s\rangle_j = s|s\rangle_j, \tag{12}$$

where $|s\rangle_j$ denotes the jth spin-s highest state vector.

It turns out that this is one possible eigenvector on which the matrix elements of $\mathcal{T}_{p,q}(\lambda)$ operate as follows,

$$\mathcal{T}_{p,q}(\lambda)|0\rangle = \begin{cases} \prod_{j=1}^{N} S(\lambda, \mu_j)_{p,1}^{p,1}|0\rangle & \text{for } p = q \\ 0 & \text{for } p < q \\ |pq\rangle & \text{for } p < q \end{cases} \tag{13}$$

where $|pq\rangle$ are non-null vectors representing excitations over the vacuum $|0\rangle$.

From the above expressions we see that the fields $\mathcal{T}_{p,q}(\lambda)$ for $p > q$ act as creation operators with respect to the reference state $|0\rangle$. Of particular importance are the operators $\mathcal{T}_{1,q}(\lambda)$ $(q \geqslant 2)$ which satisfy the following property,

$$\left[\mathcal{T}_{1,q}(\lambda), \sum_{j=1}^{N} S_j^z\right] = (q-1)\mathcal{T}_{1,q}(\lambda). \tag{14}$$

A direct consequence of (14) is that the fields $\mathcal{T}_{1,q}(\lambda)$ can be interpreted as raising operators associated with excitations over the ferromagnetic vacuum $|0\rangle$ with spin component $s-q+1$. It is therefore plausible to suppose that the eigenvectors $|\psi\rangle = \Phi_m(\lambda_1, \ldots, \lambda_m)|0\rangle$ of (8) should be constructed similarly as a Fock space having $2s$ possible distinct creation fields. Considering that both the total number of particles and spin are conserved quantities one expects that the eigenvectors structure should be as follows. The field $\mathcal{T}_{1,2}(\lambda_1)$ represents the one-particle state, the linear combination $\mathcal{T}_{1,2}(\lambda_1)\mathcal{T}_{1,2}(\lambda_2) + \psi_1(\lambda_1, \lambda_2)\mathcal{T}_{1,3}(\lambda_1)\mathcal{T}_{1,1}(\lambda_2)$, for some function $\psi_1(\lambda_1, \lambda_2)$, the two-particle states and so forth. It turns out that the form of such linear combinations can be inferred on the basis of the commutation rules between the fields $\mathcal{T}_{1,q}(\lambda)$ derived from the quadratic algebra (11). In addition to that, we emphasize that much of the simplifications needed to construct suitable eigenstates are carried out only on the basis of an extensive use of the Yang–Baxter (4) and the unitarity (5) constraints between the scattering amplitudes. Omitting here the technicalities of these computations [18] we find that the multi-particle states obey a $2s$-order recursion relation given by,

$$\begin{aligned} \Phi_m(\lambda_1, \ldots, \lambda_m) = &\sum_{k=1}^{\min\{2s,m\}} \sum_{2\leqslant j_2<\cdots<j_k\leqslant m} \psi_{\frac{k}{2}}\left(\lambda_1, \lambda_{j_2}, \ldots, \lambda_{j_k}\right)\mathcal{T}_{1,1+k}(\lambda_1)\Phi_{m-k} \\ &\times (\lambda_2, \ldots, \lambda_{j_2-1}, \lambda_{j_2+1}, \ldots, \lambda_{j_3-1}, \lambda_{j_3+1}, \ldots, \lambda_{j_k-1}, \lambda_{j_k+1}, \ldots, \lambda_m) \\ &\times \prod_{d=2}^{k} \mathcal{T}_{1,1}\left(\lambda_{j_d}\right) \prod_{l=2}^{k} \prod_{\substack{t_1=2 \\ t_1\neq j_2,\ldots,j_k}}^{m} \frac{S\left(\lambda_{t_1}, \lambda_{j_l}\right)_{1,1}^{1,1}}{S\left(\lambda_{t_1}, \lambda_{j_l}\right)_{2,1}^{2,1}} \prod_{\substack{t_2=2 \\ t_2\neq j_2,\ldots,j_k}}^{j_l} \Theta\left(\lambda_{t_2}, \lambda_{j_l}\right), \end{aligned} \tag{15}$$

where we identify $\Phi_0 \equiv 1$ and function $\Theta(\lambda, \mu)$ is[1],

$$\Theta(\lambda, \mu) = \frac{S(\lambda, \mu)_{3,1}^{3,1} S(\lambda, \mu)_{2,2}^{2,2} - S(\lambda, \mu)_{3,1}^{2,2} S(\lambda, \mu)_{2,2}^{3,1}}{S(\lambda, \mu)_{1,1}^{1,1} S(\lambda, \mu)_{3,1}^{3,1}}. \tag{16}$$

In the course of our analysis we also have made the natural hypothesis that the field states (15) are symmetric functions in all $\{\lambda_j\}$ variables. This exchange property between the variables λ_1 and λ_2 can be used to determine $\psi_{\frac{k}{2}}\left(\lambda_1, \lambda_{j_2}, \ldots, \lambda_{j_k}\right)$ in terms of the scattering

[1] We remark that for $s = \frac{1}{2}$ both numerator and denominator of (16) vanish and the respective limit gives us $\Theta(\lambda, \mu) = \frac{S(\lambda,\mu)_{2,2}^{2,2}}{S(\lambda,\mu)_{1,1}^{1,1}}$.

amplitudes recursively. The simplest case being $s = 1/2$ where by fixing $\psi_{\frac{1}{2}}(\lambda) = 1$ we find that $\psi_1(\lambda, \mu)$ is

$$\psi_1(\lambda, \mu) = -\frac{S(\lambda, \mu)_{3,1}^{2,2}}{S(\lambda, \mu)_{3,1}^{3,1}}. \tag{17}$$

Similar task for higher spin is in general more involved but it can be done for any specific value of the spin [18]. As an extra example we present below the explicit expression for $s = 3/2$,

$$\begin{aligned}\psi_{\frac{3}{2}}(\lambda, \mu, \tau) = {}& \frac{\Theta(\lambda, \mu)\psi_1(\lambda, \tau)S(\lambda, \mu)_{1,1}^{1,1}S(\lambda, \mu)_{4,1}^{3,2}}{S(\lambda, \mu)_{3,2}^{4,1}S(\lambda, \mu)_{4,1}^{3,2} - S(\lambda, \mu)_{3,2}^{3,2}S(\lambda, \mu)_{4,1}^{4,1}} \\ & + \psi_1(\mu, \tau)\frac{S(\lambda, \mu)_{3,2}^{2,3}S(\lambda, \mu)_{4,1}^{3,2} - S(\lambda, \mu)_{3,2}^{3,2}S(\lambda, \mu)_{4,1}^{2,3}}{S(\lambda, \mu)_{3,2}^{4,1}S(\lambda, \mu)_{4,1}^{3,2} - S(\lambda, \mu)_{3,2}^{3,2}S(\lambda, \mu)_{4,1}^{4,1}}.\end{aligned} \tag{18}$$

For $s = 1/2$ our result (15) recovers the known algebraic Bethe states associated with the six-vertex model [9, 10] while for $s = 1$ we have an extension of the Bethe states for nineteen-vertex models [19]. In fact, apart from factorizability and unitarity, no other assumptions on the S-matrices elements entering the eigenstates (15) have been made. In addition, the generality of our recursive manner to generate the eigenstates for arbitrary spin-s is, as far as we know, a new progress in the quantum inverse scattering approach. It follows from the commutation relation (14) and our construction (15) the property,

$$\sum_{j=1}^{N} S_j^z \Phi_m(\lambda_1, \ldots, \lambda_m)|0\rangle = (sN - m)\Phi_m(\lambda_1, \ldots, \lambda_m)|0\rangle, \tag{19}$$

which corroborates the physical interpretation of $\Phi_m(\lambda_1, \ldots, \lambda_m)|0\rangle$ being multi-particle states over $|0\rangle$.

We now turn our attention to the determination of the eigenvalues $\Lambda(\lambda, \vec{\mu})$. From (8) one has to carry on the diagonal fields $\mathcal{T}_{p,p}(\lambda)$ over the creation fields $\mathcal{T}_{1,q}(\mu)$ $(q \geqslant 2)$ that built the multi-particle states (15). This task is made by recasting the Yang–Baxter algebra (11) in the form of commutation rules between these fields. In general, convenient commutation rules do not follow immediately from (11) and a two-step procedure is needed. To give an example of our approach let us denote by $[l; k]$ the lth row and the kth column of (11). The appropriate commutation relations between the fields $\mathcal{T}_{1,2}(\lambda)$ and $\mathcal{T}_{p,p}(\lambda)$ for $1 < p < n_s$ are obtained by using the combination $[l; (l-1)*n_s + 2]S_{l+1,1}^{l+1,1}(\lambda, \mu) - [l; l*n_s + 1]S_{l+1,1}^{l,2}(\lambda, \mu)$ of the entries of (11). The basic idea is to keep the diagonal operator $\mathcal{T}_{p,p}(\lambda)$ always in the right-hand side position in the commutation rules and such procedure can be implemented for all creation fields $\mathcal{T}_{1,q}(\mu)$. The eigenvalues are easily collected by keeping only the first terms of the commutation rules among $\mathcal{T}_{1,q}(\mu)$ and $\mathcal{T}_{p,p}(\lambda)$ and after some cumbersome simplifications we find that the final result for $\Lambda(\lambda, \vec{\mu})$ is,

$$\begin{aligned}\Lambda(\lambda, \vec{\mu}) = {}& \prod_{i=1}^{N} S(\lambda, \mu_i)_{1,1}^{1,1} \prod_{l=1}^{m} \frac{S(\lambda_l, \lambda)_{1,1}^{1,1}}{S(\lambda_l, \lambda)_{2,1}^{2,1}} + \sum_{k=2}^{n_s-1} \prod_{i=1}^{N} S(\lambda, \mu_i)_{k,1}^{k,1} \prod_{l=1}^{m} F_k(\lambda, \lambda_l) \\ & + \prod_{i=1}^{N} S(\lambda, \mu_i)_{n_s,1}^{n_s,1} \prod_{l=1}^{m} \frac{S(\lambda, \lambda_l)_{n_s,2}^{n_s,2}}{S(\lambda, \lambda_l)_{n_s,1}^{n_s,1}},\end{aligned} \tag{20}$$

where

$$F_k(\lambda, \mu) = \frac{S(\lambda, \mu)_{k,2}^{k,2}S(\lambda, \mu)_{k+1,1}^{k+1,1} - S(\lambda, \mu)_{k,2}^{k+1,1}S(\lambda, \mu)_{k+1,1}^{k,2}}{S(\lambda, \mu)_{k,1}^{k,1}S(\lambda, \mu)_{k+1,1}^{k+1,1}}. \tag{21}$$

The remaining terms that are not proportional to the eigenvector (15) can be cancelled out by imposing further restriction on the rapidities $\{\lambda_j\}$. These are auxiliary Bethe ansatz equations and in our case only a unique relation is needed to eliminate all such unwanted terms. It is given by

$$\prod_{l=1}^{N} \frac{S(\lambda_j, \mu_l)_{1,1}^{1,1}}{S(\lambda_j, \mu_l)_{2,1}^{2,1}} = \prod_{\substack{i=1\\ i\neq j}}^{m} \Theta(\lambda_j, \lambda_i) \frac{S(\lambda_j, \lambda_i)_{1,1}^{1,1}}{S(\lambda_j, \lambda_i)_{2,1}^{2,1}} \frac{S(\lambda_i, \lambda_j)_{2,1}^{2,1}}{S(\lambda_i, \lambda_j)_{1,1}^{1,1}}. \tag{22}$$

Before proceeding we remark that our expressions (20)–(22) reproduce the eigenvalues and Bethe ansatz equations of a particular integrable model with higher spin obtained through fusion procedure of six-vertex S-matrices [12]. Even in this special case, one hopes that our general expression for the eigenstates (15) could still be of utility as far as correlation functions are concerned [20]. We note that a systematic classification of the solutions of the Yang–Baxter equation (4) is beyond the reach at present. We expect therefore that our results will be useful to other $U(1)$ S-matrices, specially because no assumption on their spectral parameter dependence has been made.

At this point we have been able to derive that basic ingredients to obtain the quantization rule for the one-particle momenta $p(\mu_j)$. This follows directly from (6), (20) and the unitarity property of the S-matrix (5), i.e. the condition $S(\lambda, \lambda)_{a,b}^{c,d} \sim \delta_{a,d}\delta_{b,c}$. The final result is

$$\mathrm{e}^{\mathrm{i}k(\mu_j)L} = \prod_{\substack{i=1\\ i\neq j}}^{N} S(\mu_j, \mu_i)_{1,1}^{1,1} \prod_{l=1}^{m} \frac{S(\lambda_l, \mu_j)_{1,1}^{1,1}}{S(\lambda_l, \mu_j)_{2,1}^{2,1}}. \tag{23}$$

The results (22), (23) are essential in order to investigate the thermodynamic limit properties for a fixed density N/L. They have the merit of being derived under mild assumptions for the two-body collision amplitudes and therefore with potential for widespread applicability.

In conclusion, we have solved exactly the eigenspectrum of a system of spin-s particles that interact via arbitrary $U(1)$ factorizable S-matrix. The structure of both eigenvectors and the eigenvalues was derived solely from the Yang–Baxter relations (4), (11) and the unitarity property (5). We believe that our formula (15) is capable of accommodating the solution of other integrable models possessing extra $U(1)$ symmetries other than that already discussed. In these cases, previous experience [21] suggests that the recursive structure of (15) will be preserved but now with $\psi_{\frac{k}{2}}\left(\lambda_1, \lambda_{j_2}, \ldots, \lambda_{j_k}\right)$ behaving as a vector function while $\Theta(\lambda, \mu)$ as an underlying factorizable S-matrix. If this proposal turns out to be feasible in the future one would be able to solve exactly several distinct families of integrable models from a rather unified point of view.

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